

# Coexistence of bulk antiferromagnetic order and superconductivity in the QED<sub>3</sub> theory of copper oxides

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(Dated: February 1, 2008)

Within the framework of the QED<sub>3</sub> theory of cuprate superconductivity it is argued that bulk antiferromagnetic (AF) order can coexist with *d*-wave superconductivity in the underdoped region, in agreement with recent experiments. The AF order arises from the phase fluctuating *d*-wave superconductor via the mechanism of spontaneous chiral symmetry breaking, provided that fluctuations are sufficiently strong. The phase diagram for this coexistence is mapped out by means of analytical and numerical solutions of the underlying Dyson-Schwinger equation in the large  $N$  limit.

Interplay between the AF order and superconductivity remains one of the central themes in the physics of the high- $T_c$  cuprates [1, 2, 3, 4]. Recently, an appealing new connection between AF and *d*-wave superconducting (*d*SC) orders has been pioneered, based on the ideas originally articulated by Emery and Kivelson [5]. When long range *d*SC order is destroyed by thermal or quantum vortex-antivortex fluctuations [6, 7, 8, 9, 10], the resulting state can be either a symmetric algebraic Fermi liquid [11] or, if the fluctuations are sufficiently strong, an *incommensurate antiferromagnet* [12, 13, 14] with ordering vector  $\mathbf{Q}$  illustrated in Fig. 1. In the latter case AF order arises through an inherent dynamical instability of the underlying effective low energy theory of a phase fluctuating *d*-wave superconductor, a (2+1) dimensional quantum electrodynamics, QED<sub>3</sub> [10, 11]. This instability is known as the spontaneous chiral symmetry breaking (CSB) [15, 16, 17] and is a well studied phenomenon in the particle physics.

In this communication we investigate the possibility that *bulk* AF order can set in within the superconducting phase, resulting in the region of coexistence of AF and *d*SC orders in the phase diagram shown in Fig. 1, within the QED<sub>3</sub> framework. Other approaches to this problem have been considered previously; see e.g. Ref. [18]. Recently, experiments have found tantalizing hints of such coexistence in zero applied magnetic field in Y and La based cuprates [19, 20, 21, 22, 23]. It has been shown previously [24] that within the QED<sub>3</sub> framework such coexistence indeed can occur *locally* in the vicinity of fluctuating field-induced vortices, as found in neutron scattering [25, 26],  $\mu$ SR [27] and STM [28] experiments.

Within the QED<sub>3</sub> theory [10] the dynamical agent responsible for the emergence of AF order is a noncompact U(1) Berry gauge field  $a_\mu$  which encodes the topological frustration encountered by the nodal fermions as they propagate on the background of the fluctuating vortex-antivortex plasma. In the non-superconducting phase Berry gauge field is *massless* [10, 11, 12]. Its quanta, “berrions”, have the same effect as photons in ordinary QED<sub>3</sub>: they mediate long range interactions between fermions and precipitate the chiral instability if the number of fermion species  $N$  (pairs of Dirac nodes per unit cell in a *d*SC) is less than a critical value  $N_c \simeq 3.2$  [15, 16].

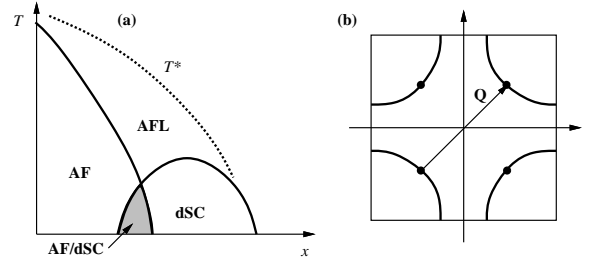


FIG. 1: a) Schematic phase diagram of a cuprate superconductor with AF/*d*SC coexistence. AFL denotes algebraic Fermi liquid, the symmetric phase of QED<sub>3</sub>. b) Cuprate Fermi surface with internodal wavevector  $\mathbf{Q}$ .

In the *d*SC phase, as vortices bind to finite pairs or loops, the Berry gauge field becomes *massive* [10, 11, 12]. Nominally, one would expect that coupling to such massive gauge field should become irrelevant for the low energy physics. Here we show that this is not necessarily so. Based on the approximate analytical and full numerical solutions of the underlying large- $N$  Dyson-Schwinger equation for the fermion self energy we find that solutions with broken chiral symmetry and finite fermion mass  $m_D$  can be found even when the gauge field acquires small mass  $m_a$ . This opens up a possibility for the coexistence of bulk AF and *d*SC orders in the QED<sub>3</sub> theory of underdoped cuprates.

As shown in Refs. [10, 11] the low-energy dynamics of fermionic quasiparticles in a *d*-wave superconductor coupled to fluctuating vortices is governed by the QED<sub>3</sub> Euclidean action  $S = \int d^3x \mathcal{L}_D$  with

$$\mathcal{L}_D \equiv \sum_{l=1}^N \bar{\Psi}_l(x) \gamma_\mu (i\partial_\mu - a_\mu) \Psi_l(x) + \mathcal{L}_B[a(x)]. \quad (1)$$

Here,  $\Psi_l(x)$  is a four component Dirac spinor representing the “topological” fermion excitations associated with a pair of antipodal nodes,  $x = (\tau, \mathbf{r})$  denotes the space-time coordinate, and  $\gamma_\mu$  ( $\mu = 0, 1, 2$ ) are the gamma matrices satisfying  $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ . The number  $N$  of fermion species is equal to  $2n_{\text{CuO}}$ , with  $n_{\text{CuO}}$  denoting the number of CuO planes per unit cell [11]. Lagrangian  $\mathcal{L}_B$  encodes the dynamics of the gauge field  $a_\mu$  and is

given by  $\mathcal{L}_B[a] = \frac{1}{2}\Pi_{\mu\nu}(q)a_\mu(q)a_\nu(-q)$  with

$$\Pi_{\mu\nu}(q) = \frac{1}{e^2} (m_a^2 + \alpha|q| + q^2) \left( \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right). \quad (2)$$

The mass and Maxwell terms in Eq. (2) originate from the bare vortex action;  $m_a = 0$  in the non-superconducting phase, reflecting the unbound nature of vortex-antivortex fluctuations, while  $m_a > 0$  in the superconductor [10, 11, 12]. The  $q$ -linear term, with  $\alpha = Ne^2/8$ , reflects the partial screening of the gauge field by the medium of topological fermions, expressed to one loop order.

We analyze the CSB phenomenon in the Lagrangian (1,2) by solving the Dyson-Schwinger (DS) equation for the fermion self energy  $\Sigma(p)$  to leading order in  $1/N$  expansion. This technique is known to be robust against the higher order  $1/N$  corrections even for small  $N \sim 2$  [17, 29], and the results appear to be insensitive to vertex corrections [30].

With the above definitions we can write the DS equation for the fermion self energy as [16]

$$\Sigma(p) = \frac{8\alpha}{N} \int \frac{d^3k}{(2\pi)^3} \frac{\gamma_\mu D_{\mu\nu}(p-k)\Sigma(k)\gamma_\nu}{k^2 + \Sigma^2(k)}, \quad (3)$$

where  $D_{\mu\nu}(q) = \Pi_{\mu\nu}^{-1}(q)$  is the gauge field propagator in the Landau gauge. Following Pisarski [15], the simplest way to establish the spontaneous mass generation for fermions is to assume that, crudely,  $\Sigma(p)$  is constant over most of the integration range,  $\Sigma(p) = m_D$ , and demand self consistency for the constant value  $m_D$ :

$$m_D = \frac{8\alpha}{N} \int \frac{d^3k}{(2\pi)^3} \frac{\gamma_\mu D_{\mu\nu}(k)m_D\gamma_\nu}{k^2 + m_D^2}. \quad (4)$$

The above DS equation always has a trivial solution  $m_D = 0$ , corresponding to the chirally symmetric state. A non-trivial, symmetry broken solution  $m_D > 0$  satisfies

$$1 = \frac{8\alpha}{\pi^2 N} \int_0^\infty \frac{k^2 dk}{(k^2 + m_D^2)(m_a^2 + \alpha k + k^2)}, \quad (5)$$

where we have taken the trace of both sides and performed the angular integrals. For small gauge field mass,  $m_a^2 \ll \alpha m_D$ , we observe that the fermion mass effectively cuts off the integral in the infrared while  $\alpha$  plays the role of the ultraviolet cutoff. We may thus approximate Eq. (5) by

$$1 = \frac{8\alpha}{\pi^2 N} \int_{m_D}^\alpha \frac{k^2 dk}{k^2(\alpha k + m_a^2)} = \frac{8}{\pi^2 N} \ln \left( \frac{\alpha^2}{\alpha m_D + m_a^2} \right).$$

In this approximation the fermion mass is given by

$$m_D = \alpha e^{-\pi^2 N/8} - \frac{m_a^2}{\alpha}. \quad (6)$$

This is just the classic result of Pisarski [15] with a small correction due to the gauge field being massive. Eq. (6)

thus suggests that CSB in QED<sub>3</sub> could survive in the presence of small gauge field mass. Translated into the language of condensed matter physics Eq. (6) suggests that even *bound* vortex-antivortex fluctuations could give rise to AF instability *within* the state with the true  $d$ SC long range order.

One can also perform the integral appearing in Eq. (5) exactly and confirm that the solution (6) indeed emerges in the limit  $m_a^2 \ll \alpha m_D$  (up to an unimportant prefactor). The approximation employed to obtain Eq. (6) is instructive in that it reveals the fundamental reason for the insensitivity of CSB to small gauge field mass: the largest contribution to the RHS of Eq. (5) comes from the region  $m_D < k < \alpha$ . Therefore, modifying the gauge boson propagator at momenta  $k \ll m_D$  by introducing the mass term has very little effect on the system.

The above solution is highly suggestive but because of the crude approximations involved it is deficient in at least two ways. First, for  $m_a = 0$ , it fails to reproduce the result of a more refined treatment of DS Eq. (3) [16, 17] that no CSB occurs for  $N > N_c$ , with  $N_c = 32/\pi^2$  to leading order in  $1/N$  [16]. Second, Eq. (6) is only valid for  $m_a^2 \ll \alpha m_D$ . In what follows we present a more careful treatment of Eq. (3) which is free of the above deficiencies. We use these solutions to map out the phase diagram of CSB as a function of  $N$  and  $m_a$ .

To this end we must relax our assumption of constant  $\Sigma(p)$  in Eq. (3) and treat its full functional dependence on the three-momentum  $p$ . It is still possible to perform the angular integral to obtain

$$\begin{aligned} \Sigma(p) &= \frac{4\alpha}{\pi^2 N p} \frac{1}{\sqrt{\alpha^2 - 4m_a^2}} \int_0^\infty dk \frac{k \Sigma(k)}{k^2 + \Sigma^2(k)} \\ &\times \left[ \theta_1 \ln \left( \frac{k+p+\theta_1}{|k-p|+\theta_1} \right) - \theta_2 \ln \left( \frac{k+p+\theta_2}{|k-p|+\theta_2} \right) \right] \end{aligned} \quad (7)$$

where  $\theta_{1,2} = \frac{1}{2}(\alpha \pm \sqrt{\alpha^2 - 4m_a^2})$ . In the limit  $\alpha^2 \gg m_a^2$  we recover the equation studied in Ref. [16],

$$\Sigma(p) = \frac{8}{\pi^2 N p} \int_{\frac{m_a^2}{\alpha}}^\infty dk \frac{k \Sigma(k)}{k^2 + \Sigma^2(k)} \min(k, p), \quad (8)$$

but with modified lower bound on the integral. In the massless theory ( $m_a = 0$ ), Eq. (8) is solved by linearizing the integrand in  $\Sigma$  and seeking the solution in the form of a power law,  $\Sigma(p) \sim p^b$ . Only solutions with complex  $b$  turn out to be physically admissible placing a condition on the number of fermion species  $N < N_c = 32/\pi^2$ . The chiral mass is then given by [16]

$$m_D \equiv \Sigma(0) = c\alpha e^{-2\pi/\sqrt{N_c/N-1}}, \quad (9)$$

with  $c$  a numerical constant.

Returning to the theory with massive gauge field, we observe that a simple power law ansatz no longer solves the linearized integral equation (8), essentially because of the appearance of the nonzero lower bound of the integral. Consequently, it is a non-trivial issue to extend

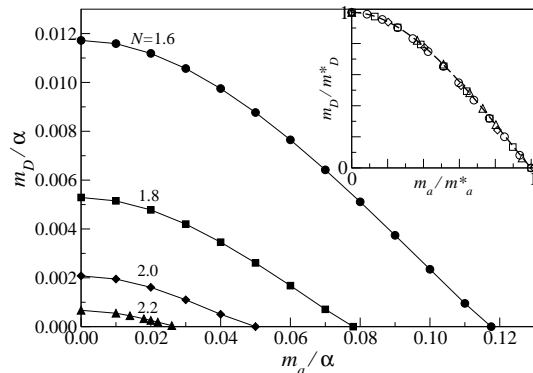


FIG. 2: Numerical solution of Eq. (7) for dynamically generated fermion mass  $m_D$  as a function of gauge field mass  $m_a$  for selected values of  $N < N_c$ . Inset: curves for various  $N$  scaled as described in the text. Dashed line represents the solution to Eq. (10).

such an analysis to our case of interest with  $m_a > 0$  [31]. Fortunately, it is relatively straightforward to analyze the full nonlinear, angle integrated, DS Eq. (7) numerically. We start from a constant  $\Sigma(k)$  and numerically evaluate the RHS at  $n$  discrete points  $p_i$ . We then iterate this procedure until the solution for  $\Sigma(p)$  no longer changes appreciably between iterations. In this we adopt an upper cutoff,  $\Lambda$ , for the integral. We find that when  $\Lambda \gtrsim \alpha$ , our solutions no longer depend on  $\Lambda$ .

Figures 2 and 3 summarize our findings. Fig. 2 displays the dependence of chiral mass  $m_D$  on the gauge field mass  $m_a$  for selected values of  $N < N_c$ . A notable feature here is the surprising resilience of the chiral mass:  $m_D$  persists until very large values of  $m_a$ . A hint of why this may be so is offered by a simple power counting in Eq. (5). This reveals that the relevant scale actually is not  $m_a$  itself but  $m_a^2/\alpha$ , with  $m_a/\alpha \ll 1$ . The latter quantity also enters our estimate for  $m_D$ , Eq. (6), which Fig. 2 confirms to be valid for small  $m_a$ .

When scaled by  $m_D^*$  and  $m_a^*$ , defined as endpoints of the curves in Fig. 2, the points for different  $N$  fall on the same universal curve (see inset). One way to understand this universality is from Eq. (5): its *exact* solution is given by the implicit equation

$$f\left(\frac{m_D}{m_D^*}, \frac{m_a^2}{m_a^{*2}}\right) = 0, \quad (10)$$

with  $f(x, y) = x^2 \ln x + y^2 \ln y + \frac{\pi}{2}xy$ , independent of  $N$ . This solution is displayed as a dashed line in the inset of Fig. (2). We find that the scaling holds very well up to  $N \approx 2.9$ ; closer to  $N_c^0 \equiv N_c(m_a = 0) = 32/\pi^2$  the agreement gets progressively worse. This comparison indicates that the approximation of constant self energy, leading to Eq. (5), is very accurate for determining  $m_D$ , as long as  $N$  is not too close to  $N_c^0$ .

Another way of viewing the above data is to consider  $N_c$  as a function of  $m_a$ . Fig. 3 shows this dependence as determined by our numerical solution of Eq. (7). As

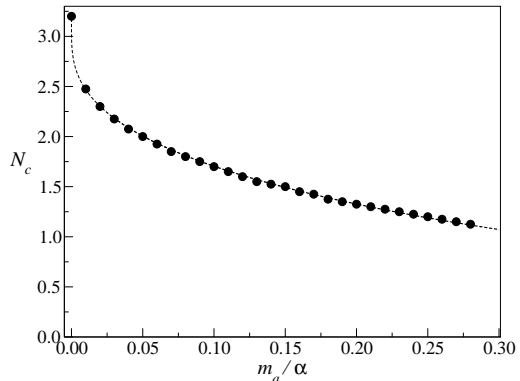


FIG. 3: Numerical solution of Eq. (7) for  $N_c$  as a function of gauge field mass  $m_a$ . The dashed line represents a fit to the heuristic formula (12) with  $C = 2.7$ .

expected  $N_c$  is a decreasing function of  $m_a$ . A striking feature is that  $N_c$  remains relatively large for  $m_a$  up to significant fraction of  $\alpha$ . The following simple argument leads to a highly accurate heuristic formula for  $N_c(m_a)$ . At low energies there are only two scales in the problem:  $m_D$  and  $m_a$ . Increasing  $m_a$  leads to reduction of  $m_D$ . It is then natural to expect that  $m_D$  vanishes when the two scales cross, i.e.  $m_a \approx m_D^*$ . Here  $m_D^*$  is the fermion mass for given  $N$  but with massless gauge field, given by Eq. (9). Given our previous insight that the relevant quantity is  $m_a^2/\alpha$  we take our criterion to read

$$(m_a/\alpha)^2 = C(m_D^*/\alpha), \quad (11)$$

with  $C$  is a numerical constant. Inserting  $m_D^*$  from Eq. (9) and solving for  $N$  we obtain

$$N_c(m_a) = \frac{32}{\pi^2} \left[ 1 + \left( \pi / \ln \frac{C\alpha}{m_a} \right)^2 \right]^{-1}, \quad (12)$$

where  $C = \sqrt{eC}$ . As seen from the fit in Fig. 3 for  $C \approx 2.7$  this leads to an excellent agreement with our numerical results. We note that the same expression, with  $C = e^2 \approx 7.4$ , was obtained by Gusynin *et al.* [31] from the linearized version of the DS equation with an infrared cutoff.

Our results presented above firmly establish the existence of nontrivial, chiral symmetry broken solution of the QED<sub>3</sub> Dyson-Schwinger equation in the presence of a small gauge field mass. The question arises whether this represents a true ground state of the system, as is known to be the case for the massless-photon QED<sub>3</sub> [16]. We believe this remains true when  $m_a > 0$ . Imagine turning on the gauge field mass from zero deep inside the symmetry broken phase, where there is large energy difference between the ground state and the “false vacuum” represented by chirally symmetric phase. It is then clear that introduction of arbitrarily small gauge field mass cannot raise the energy of the ground state above that of the false vacuum: the chirally broken phase must remain a ground

state of the system for some range of values  $m_a < m_a^*$ , as indicated by our solutions (6) and Fig. 2.

Detailed nature of the criticality in the presence of gauge field mass appears to be a nontrivial issue. The transition in the ordinary massless-photon QED<sub>3</sub> is thought to be of a special “conformal” variety, i.e. a continuous transition of infinite order [33, 34]. This is due to the long range nature of interactions mediated by the massless gauge field. One would therefore expect that in the massive case the transition becomes a conventional second order [31, 32].

Recently, powerful field theoretic duality arguments have been advanced [35] leading to a proposal for an upper bound  $N_c^0 \leq \frac{3}{2}$ . This suggestion finds some support in numerical simulations of noncompact lattice QED<sub>3</sub> [36] which found no decisive signal for chiral mass generation at  $N = 2$ . If these results are correct this would suggest that the conventional state of the art analysis based on the Dyson-Schwinger equation (3) overestimates  $N_c^0$  by more than a factor of 2. Our analysis, based on this same technique, would then also be quantitatively inaccurate. However, we expect its qualitative features to remain valid.

If we stipulate that chiral symmetry breaking occurs in the massive-photon QED<sub>3</sub>, does this necessarily imply coexistence of the AF and *d*SC orders in cuprates? Answering this question involves additional subtlety that is related to the detailed nature of the criticality at the transition to the *d*SC state. According to the standard phenomenology [10, 11, 12, 13], approaching the transition from the above (i.e. from the symmetric pseudo-

gap phase) berryon is massless but the charge  $e$  tends to zero, transforming eventually the Maxwell term into the mass term in the *d*SC phase. Since according to Eq. (9)  $m_D$  is proportional to  $e^2$  (through  $\alpha$ ) this would suggest that chiral mass should vanish at the transition. However, there is presumably higher order ( $\sim q^4$ ) term in the bare berryon action whose prefactor also diverges at the transition giving rise to a Maxwell term in the *d*SC phase. The chiral mass would then be proportional to this new “charge” in the *d*SC phase. Our calculations above strongly suggest that such a chirally broken phase can be a globally stable ground state of a system with massive berryons. Again, while this work might have resolved the nature of the phases on both sides of the transition, the detailed understanding of the criticality itself appears to be a complex problem awaiting future resolution.

QED<sub>3</sub> theory as formulated in Refs. [10, 11] therefore appears to support a locus of AF/*d*SC coexistence in the low doping region of the phase diagram Fig. 1. In this region the system remains superconducting while fermionic excitations become fully gapped. This small gap should be observable in thermodynamic and transport measurements. In particular the superfluid density should become exponentially activated in very clean samples at low temperatures.

The authors are indebted to I. F. Herbut, D. E. Sheehy and Z. Tešanović for many penetrating discussions. This work was supported by NSERC; in addition M.F. acknowledges the support of the A. P. Sloan Foundation.

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- [1] P.W. Anderson, *The theory of superconductivity in the high- $T_c$  cuprates*, Princeton University Press (1997).
  - [2] S.C. Zhang, *Science* **275**, 1089 (1997).
  - [3] P. A. Lee, N. Nagaosa, T. K. Ng, and X. -G. Wen, *Phys. Rev. B* **57**, 6003 (1998).
  - [4] M. Vojta and S. Sachdev, *Phys. Rev. Lett.* **83** 3916 (1999).
  - [5] V. J. Emery and S. A. Kivelson, *Nature* **374**, 434 (1995).
  - [6] J. Corson *et al.*, *Nature* **398**, 221 (1999).
  - [7] Z. A. Xu *et al.*, *Nature* **406**, 486 (2000); Y. Wang *et al.*, *Phys. Rev. Lett.* **88**, 257003 (2002).
  - [8] B. W. Hoogenboom *et al.*, *Phys. Rev. B* **62**, 9179 (2000).
  - [9] L. Balents, M.P.A. Fisher and C. Nayak, *Phys. Rev. B* **60**, 1654 (1999).
  - [10] M. Franz and Z. Tešanović, *Phys. Rev. Lett.* **87**, 257003 (2001).
  - [11] M. Franz, Z. Tešanović and O. Vafek, *Phys. Rev. B* **66**, 054535 (2002).
  - [12] I.F. Herbut, *Phys. Rev. Lett.* **88**, 047006 (2002).
  - [13] Z. Tešanović, O. Vafek and M. Franz, *Phys. Rev. B* **65** 180511 (2002).
  - [14] B. H. Seradjeh and I. F. Herbut, cond-mat/0207221.
  - [15] R.D. Pisarski *Phys. Rev. D* **29**, 2423 (1984).
  - [16] T. W. Appelquist, M. Bowick, D. Karabali and L.C.R. Wijewardhana, *Phys. Rev. D* **33**, 3704 (1986).
  - [17] D. Nash, *Phys. Rev. Lett.* **62**, 3024 (1989).
  - [18] H. Chen, Z. Su and L. Yu *Phys. Rev. B* **41**, 267 (1990).
  - [19] J. H. Brewer *et al.*, *Physica (Amsterdam)* **162C**, 157 (1989).
  - [20] Ch. Niedermayer *et al.*, *Phys. Rev. Lett.* **80**, 3843 (1998).
  - [21] J. Sonier *et al.*, *Science* **292**, 1692 (2001).
  - [22] H. A. Mook, Pengcheng Dai and F. Dogan, *Phys. Rev. B* **64**, 012502 (2001).
  - [23] Y. Sidis *et al.*, *Phys. Rev. Lett.* **86**, 4100 (2001).
  - [24] M. Franz, D. E. Sheehy and Z. Tešanović, *Phys. Rev. Lett.* **88**, 257005 (2002).
  - [25] B. Lake *et al.*, *Science* **291** 1759 (2001).
  - [26] B. Lake *et al.*, *Nature* **415**, 299 (2002).
  - [27] R. I. Miller *et al.*, *Phys. Rev. Lett.* **88**, 137002 (2002).
  - [28] J.E. Hoffman *et al.*, *Science* **295**, 466 (2002).
  - [29] V. Gusynin, A. Hams and M. Reenders, *Phys. Rev. D* **63**, 045025 (2001).
  - [30] P. Maris, *Phys. Rev. D* **54**, 4049 (1996).
  - [31] V. Gusynin, M. Hashimoto, M.Tanabashi and K. Yamawaki, *Phys. Rev. D* **65**, 116008 (2002).
  - [32] V. Gusynin, *private communication*.
  - [33] T. Appelquist, J. Terning and L.C.R. Wijewardhana, *Phys. Rev. Lett.* **75**, 2081 (1995).
  - [34] V.A. Miransky and K. Yamawaki, *Phys. Rev. D* **55**, 5051 (1997).
  - [35] T. Appelquist, A. G. Cohen and M. Schmaltz, *Phys. Rev. D* **60**, 45003 (1999).

- [36] S. J. Hands, J. B. Kogut and C. G. Strouthos,  
hep-lat/0208030.